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ROBUST FIRST-ORDER EFFICIENT DESIGNS INVARIABLY APPLICABLE FOR MANY LIFETIME DISTRIBUTIONS

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Abstract: Lifetime distributions are mostly Weibull, exponential, gamma and lognormal, and these observations may be correlated. For lifetime improvement experiments, optimal settings of the operating conditions are identified using D-optimal, or rotatable designs. Therefore, for correlated lifetime observations with different distributions, locating the optimal operating settings is the primary requirement to the quality engineers. The current report derives some efficient rotatable designs for autocorrelated and a particular form of compound symmetry correlated error structures for the above mentioned four lifetime distributions. Note that the derived designs depend on the concerned correlated error structure but free of correlation coefficient values and the lifetime distributions..

Keywords: Autocorrelated errors; Compound symmetry structure; Invariably designs; Lifetime distributions; Mean lifetime model; Robust first-order rotatability.

1. Introduction

In usual response surface methodology (RSM), response distribution is assumed as normal with uncorrelated errors and equal variance (Box and Hunter 1957; Box and Draper 2007). In lifetime betterment experiments, usual RSM is adopted for searching the optimal level combinations to reach the specific target (Nair *et al.* 1992; Myers *et al.* 2002; Das and Lee 2009). A lifetime random variable commonly follows gamma, or lognormal, or exponential, or Weibull distributions (Lawless, 1982; Das 2013), and oftenly the lifetime observations may be correlated (Myers *et al.* 2002). So, the usual RSM is not appropriate for lifetime betterment experiments, as it does not meet the necessary lifetime conditions. First and second-order response surface designs with correlated errors under normal distribution have been introduced by Panda and Das (1994) and Das (1997), respectively. First-order response surface designs under correlated errors with the above four lifetime distributions have been searched by Das and Huda (2011), Das and Lin (2011), Das, Kim and Park (2015), Das, Kim and Lee (2015).

Commonly, a first-order model is an enormous approximation to the unknown true surface in a miniature experimental region of the explanatory variables y's. These models are commonly adopted when the experimenters are far away from the optimal operating process. A second-order polynomial model is frequently used for a system with curvature. Considering the lifetime x's is a larger-the-better characteristic, the steepest ascent direction is the direction in which the estimated lifetime (x's) increases. The steepest ascent method helps us for searching sequentially along the path of maximum increase in lifetime through sequentially choosing suitable response function. A lifetime experimenter is searching a process for a new region in which the lifetime of the product is improved.

Das, Kim and Lee (2015) have searched first-order lifetime designs under autocorrelated error structure with the above four distributions, and the designs are not D-optimal, and simply they are only robust rotatable but not so efficient designs. Das, Kim and Park (2015) have studied first-order D-optimal lifetime designs for compound symmetry and inter-class structure, but no author has studied lifetime betterment designs under a special form of compound symmetry structure, which is introduced herein. The present report searches some efficient first-order lifetime designs under autocorrelated error structure along with some new designs under a special form of compound symmetry structure.

The current report considers four lifetime distributions namely, gamma, exponential, Weibull and lognormal. Suppose the lifetime (T_0) follows Weibull distribution with probability density function (p.d.f.):

$$\frac{\delta}{\alpha}\left(\frac{t_0}{\alpha}\right)^{\delta-1}\exp\left[-\left(\frac{t_0}{\alpha}\right)^{\delta}\right];\ t_0 \geq 0.$$

Suppose controllable experimental variables $y = (y_1, y_2, ..., y_k)'$ which are adopted to illustrate the heterogeneity in lifetime T_0. Commonly, in Weibull regression models, it is adopted that just α, but not δ, depends on y, and α can be replaced by $\alpha(y)$ (Lawless, 1982). Hence, the p.d.f. of X_0 = $\ln T_0$, given y is as follow

$$f(x_0|y) = \frac{1}{\sigma}\exp\left[\frac{x_0-\mu(y)}{\sigma} - \exp\left(\frac{x_0-\mu(y)}{\sigma}\right)\right]; -\infty < x_0 < \infty,$$

where $\sigma = \frac{1}{\delta}$ and $\mu(y) = \ln\alpha(y)$. Form the above distribution of $X_0|\,y$, it can be written as

$$X_0 = \mu(y) + \sigma h, \tag{1.1}$$

where h follows the standard extreme value distribution with p.d.f.: $\exp(h - e^h)$; $-\infty < h < \infty$, and $E(h) = -v = 0.5772...$, known as Euler's constant, $\mathrm{Var}(h) = \pi^2/6$. Equation (1.1) is recognized as a location-scale regression model with random component h. From (1.1) a class of models can be developed for different options of $\mu(y) = \ln\alpha(y)$.

If $\delta = 1$, $\sigma = 1$, the stated above Weibull distribution changes to exponential lifetime distribution, and equation (1.1) changes to

$$X_0 = \mu(y) + h. \tag{1.2}$$

Suppose lifetime (T_0) follows gamma distribution with p.d.f.: , the $\frac{1}{\Gamma p}\frac{t_0^{p-1}}{\{\alpha(y)\}^p}\exp\left[-\frac{t_0}{\alpha(y)}\right]$; $t_0 \geq 0$, the p.d.f. of $X_0 = \ln T_0$, given y, is presented by

$$f(x_0|y) = \frac{1}{\Gamma p}\exp[(x_0 - \mu(y))p]\exp[-\exp(x_0 - \mu(y))]; -\infty < x_0 < \infty.$$

From the above distribution, it can be written as

$$X_0 = \mu(y) + h_1, \tag{1.3}$$

where h_1 follows log-gamma distribution with p.d.f.: $\frac{1}{\Gamma p}\exp(h_1 p - e^{h_1}); -\infty < h_1 < \infty.$

Suppose the lifetime (T_0) follows lognormal distribution (*i.e.*, $\ln T_0 (= X_0)$ follows normal distribution with mean $\ln\alpha(y)$ and variance δ_1^2) with p.d.f.: $\frac{1}{t_0\delta_1(2\pi)^{1/2}}\exp\left[-\frac{(\ln t_0 - \ln\alpha(y))^2}{2\delta_1^2}\right]$; $t_0 \geq 0$, $\delta_1 \geq 0$, then

$$X_0 = \mu(y) + \delta_1\tau, \tag{1.4}$$

where τ follows the standard normal distribution.

The importance of lifetime designs are clearly discussed using a motivating example in the articles by Das, Kim and Park (2015); Das, Kim and Lee (2015). The report is organized as follows. The subsequent section presents first-order correlated lifetime models along with first-order rotatability conditions. Section 3 presents some efficient first-order rotatable designs under autocorrelated error structure, section 4 presents robust first-order lifetime designs under a special form of compound symmetry structure, and and followed by conclusion. The fundamental contributions of the current article are given in Sections 3 and 4, which are completely new in the lifetime betterment experiments literature.

2. First-order Correlated Lifetime Models and Rotatability Conditions

2.1. First-order correlated lifetime models

For a known, or a miniature experimental region, or if the experimenters are outlying from the optimal process operating conditions, first-order models are often adopted. It is clear that the above stated four equations ((1.1) to (1.4)) have two portions namely $\mu(y)$ (systematic) and a random portion (σh, or h, or h_1, or $\delta_1\tau$) (related with the lifetime distribution). Note that the random portion distribution is fully different from the lifetime (T_0) distribution, and it is not a usual 'random effects' that are often used in random effects models namely missing information, unknown future prediction, unobserved effect, censored data etc. (Lee *et al.* 2017). First-order correlated lifetime models with gamma, or exponential, or lognormal, or Weibull distributions are derived as follows.

From (1.1) the lifetime T_0 follows Weibull distribution with mean $\alpha(y)$ (= $e^{g(y,\beta)}$, say as $t_0 > 0$) implying $\mu(y) = \ln\alpha(y) = g(y, \beta)$ which turns to X_0 (= $\ln T_0|y) = g(y, \beta) + \sigma h$, where h follows standard extreme value distribution. Assuming the first-order response surface of $g(y, \beta)$ the resultant lifetime model with Weibull distribution is:

$$\ln T_0 \,|\, y = \beta_0 + \sum_{i=1}^{k} \beta_i y_i + \sigma h,$$

where h follows the standard extreme value distribution, and its mean and variance are given in (1.1). For estimating the unknown parameters $\beta_0, \beta_1, ..., \beta_k$ one needs to conduct an experiment to generate data, based on which, estimation can be done. When one performs an experiment, it issues numerous noise factors, some of them may be even unidentifiable, or undefinable. These all possible noise factors effects on the random variable '$\ln T_0$' revealed by the experimental situations is denoted by 'e', which is recognized as an experimental error. So, the above model is written as

$$z = \ln T_0 \,|\, y + e = \beta_0 + \sum_{i=1}^{k} \beta_i y_i + \sigma h + e,$$

$$\text{or } z_u = \beta_0 + \sum_{i=1}^{k} \beta_i y_{ui} + \sigma h_u + e_u; 1 \le u \le N, \tag{2.1}$$

where β_i's are unknown regression parameters and y_{ui}'s (for the u-th run of i-th factor, $1 \le u \le N$, $1 \le i \le k$) are experimental settings (or levels) which are non-stochastic, h follows standard extreme value distribution, and e_u's presents experimental errors initiated by the noise factors. There are two random variable h and e in (2.1), where h is connected to the lifetime T_0 distribution, while e is associated with the random experiment representing all unaccounted variation sources for the experimental noise factors. Note that (2.1) is a mixed linear model, while two random variables in (2.1) are not used herein as usual random effects as in generalized linear mixed models, hierarchical generalized linear models (HGLMs) or double HGLMs (Lee *et al.*, 2017). Classical, or correlated first-order response surface models do not include the random variable connected to the lifetime distribution. It is assumed herein that h_u's are all uncorrelated and h_u and e_u are uncorrelated. Similarly as usual RSM (Box and Hunter 1957), one can assume that e_u follows normal distribution. In addition, it is assumed that e_u's are not independent but correlated, as it is often observed in practice (Myers *et al.* 2002). Hence, e follows multivariate normal distribution with $E(e) = 0$, $\text{Dis}(e) = \sigma_0^2 W_0$, and rank $(W_0) = N$, where W_0 is any unknown general error variance-covariance structure.

The equation (2.1) can be written as

$$x_u = z_u + \sigma\nu = \beta_0 + \sum_{i=1}^{k} \beta_i y_{ui} + \xi_u; 1 \le u \le N, \text{ say,}$$

$$\text{or, } X = Y\beta + \xi, \tag{2.2}$$

assuming σ to be known for the time being, ν is Euler's constant (known), $\xi_u = \sigma h_u + \sigma\nu + e_u$ is the composite error (Das, Kim and Lee 2015) and $x_u = z_u + \sigma\nu$. Here $X = (x_1, x_2, ..., x_N)'$, is the origin changed recorded natural logarithm lifetime observational vector, $\beta = (\beta_0, \beta_1, ..., \beta_k)'$ is the vector of regression coefficients of order $(k + 1) \times 1$; $Y = (1 : (y_{ui}); 1 \le u \le N; 1 \le i \le k)$ is the design matrix and $\xi = (\xi_1, \xi_2, ..., \xi_N)'$. Note that $E(\xi_u) = 0$, $\text{Var}(\xi_u) = (\sigma^2\pi^2/6 + \sigma_0^2) = \sigma_1^2$ say, $\text{Cov}(\xi_u, \xi_{u'}) = \text{Cov}(e_u, e_{u'})$, $E(\xi) = 0$, $\text{Dis}(\xi) = (\sigma^2\pi^2/6)I_N + \sigma_0^2 W_0 = \sigma_1^2 W_1$ and 'W_1' is the composite correlated error structure (Das, Kim and Lee 2015), where

$$W_1 = \begin{pmatrix} 1 & \frac{Cov(e_1, e_2)}{\sigma_1^2} & \frac{Cov(e_1, e_3)}{\sigma_1^2} \cdots & \frac{Cov(e_1, e_N)}{\sigma_1^2} \\ \frac{Cov(e_2, e_1)}{\sigma_1^2} & 1 & \frac{Cov(e_2, e_3)}{\sigma_1^2} \cdots & \frac{Cov(e_2, e_N)}{\sigma_1^2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{Cov(e_N, e_1)}{\sigma_1^2} & \frac{Cov(e_N, e_2)}{\sigma_1^2} & \frac{Cov(e_N, e_3)}{\sigma_1^2} \cdots & 1 \end{pmatrix}$$

The model (2.2) under Weibull distribution assumes σ is known, but practically, it is unknown, which is substituted by its estimate $\hat{\sigma}$ that is calculated based on the original lifetime observations T_0 which follows Weibull distribution as in (1.1). For all computation, it is considered, $x_u = z_u + \hat{\sigma}\nu$, $u = 1, 2, ..., N$.

For $\delta = 1$ (concluding, $\sigma = 1$) in (1.2), the correlated first-order lifetime model for exponential distribution is exactly same as (2.2), with $x_u = z_u + \nu$, $\xi_u = h_u + \nu + e_u$, $E(\xi_u) = 0$, and $\text{Var}(\xi_u) = \pi^2/6 + \sigma_0^2 = \sigma_1^2$ (say), $u = 1, 2 ..., N$; $E(\xi) = 0$, $\text{Dis}(\xi) = (\pi^2/6)I_N + \sigma_0^2 W_0 = \sigma_1^2 W_1$, where W_1 is as in (2.2).

The correlated first-order lifetime model for gamma distribution as in (1.3) is exactly same as (2.2), with $x_u = z_u - \alpha_0$, $\xi_u = h_{1u} - \alpha_0 + e_u$, $\alpha_0 = E(h_{1u})$, $E(\xi_u) = 0$, and $\text{Var}(\xi_u) = \alpha_1^2 + \delta_1^2\, \delta_1^2$ (say), $u = 1, 2 ..., N$, $E(\xi) = 0$, $\text{Dis}(\xi) = \alpha_1^2 I_N + \sigma_0^2 W_0 = \sigma_1^2 W_1$, considering α_0 to be known for the time being, where $\sigma_1^2 = \text{Var}(h_1)$, h_1 follows log-gamma distribution as in (1.3) and W_1 is as in (2.2). Same assumptions as h_u are satisfied by h_{1u}. For all derivations with gamma distributions, it is considered that α_0 is known, but practically α_0 is unknown which is substituted by its estimate $\hat{\alpha}_0$ with the original lifetime observations T_0, which follows gamma distribution as in (1.3). We consider that $x_u = z_u - \hat{\alpha}_0$; $u = 1, 2, ..., N$.

The correlated first-order lifetime model for lognormal distribution as in (1.4) is exactly same as (2.2), with $x_u = z_u$, $\xi_u = \delta_1\tau_u + e_u$, $E(\xi_u) = 0$ and $\text{Var}(\xi_u) = \delta_1^2 + \sigma_0^2 = \sigma_1^2$ (say), $u = 1, 2 ..., N$; $E(\xi) = 0$, $\text{Dis}(\xi) = \delta_1^2 I_N + \sigma_0^2 W_0 = \sigma_1^2 W_1$, where τ follows the standard normal distribution and W_1 is as in (2.2). Similar assumptions as h_u are satisfied by τ_u.

For the above considered four lifetime distributions, the first-order correlated response surface model is exactly same as in (2.2), but only the unknown composite error (ξ) and its variance-covariance structure is different for different lifetime distributions. Also, $\text{Dis}(\xi) = \sigma_0^2 I_N$, if $\text{Cov}(e_i, e_j) = 0$ for all $i \neq j$, *i.e.*, $W_1 = I_N$. Practically, W_1 is unknown, but it is assumed known for all theoretical derivations. Generally, W_1 contains a number of unknown constants, and in the calculations which follow, the expressions for W_1 and W_1^{-1} are obtained by those after substituting the unknown constants with suitable estimates, or some assumed values.

For the above situations, appropriate selection of the experimental levels y_{ui}'s, *i.e.*, $Y = ((y_{ui}), 1 \le u \le N, 1 \le i \le k)$, the design matrix is the most important part for obtaining maximum information regarding the unknown regression parameters. Panda and Das (1994) initiated first-order RSM with correlated errors along with response normal distribution.

2.2. First-order Correlated Lifetime Models Rotatability Conditions

For the model (2.2), the best linear unbiased estimator of β, for the known W_1 and $(Y'W_1^{-1}Y)$ positive definite, is $\hat{\beta} = (Y'W_1^{-1}Y)^{-1}(Y'W_1^{-1}X)$ with $\text{Dis}(\hat{\beta}) = (Y'W_1^{-1}Y)^{-1}$.

Follwing Das, Kim and Lee (2015), the necessary and sufficient robust first-order correlated lifetime models rotatability conditions (for all values of correlation coefficient (ρ), where $\rho \in W_1$ and for all $1 \le i, j \le k$) in the model (2.2) are

(i) $v_{0j} = 1'W^{-1}y_j = 0,$

(ii) $v_{ij} = y_i'W^{-1}y_j = 0,\ i \ne j,$

(iii) $v_{ii} = y_i'W^{-1}y_i = \text{constant} = \lambda \text{ (say)} > 0.$ (2.3)

The estimated variance, that is $\text{Var}(\hat{x}_y)$ at $y = (y_1, y_2, ..., y_k)'$ is known as variance function which is given as follows,

$$Var(\hat{x}_y) = \frac{1}{v_{00}} + \frac{1}{\lambda}\sum_{i=1}^{k} y_1^2 = f(r^2), \tag{2.4}$$

where $r^2 = \sum_{i=1}^{k} y_i^2,\ v_{00} = 1'W_1^{-1}1.$

The present paper aims to derive robust efficient first-order rotatable designs for the above considered four lifetime distributions with autocorrelated and a special form of compound symmetry error structure. The developed first-order rotatable designs herein for the above four lifetime distributions are free of these distributions and also free of the values of the correlation coefficient(s). These designs depend on the error structures only. So, the derived designs are named as invariant (free of these four distributions) and robust (free of the values of correlation coefficient(s)) first-order rotatable designs. Rotatable and robust rotatable designs are defined as follows.

Definition 2.1 Rotatable design: A design is said to be rotatable if the estimated response variance of the the correlated lifetime model (2.2) at a point (y) is a function of only the distance from the design center (*i.e.* center of the co-ordinate axes) to that point.

Definition 2.2 Robust first-order rotatable design (RFORD): A design 'd' of k factors of the correlated lifetime model (2.2) which remains first-order rotatable for all the dispersion matrices belonging to a well-defined class $W_0 = \{W$ positive definite: $W_{N\times N}$ defined by a particular correlation structure possessing a definite pattern$\}$ is called a RFORD, with reference to the variance-covariance class W_0.

3. Some Efficient Correlated Lifetime Designs Under Autocorrelated Error Structure

Das, Kim and Lee (2015) have developed some first-order rotatable designs for the above considered four lifetime distributions under autocorrelated error structure $W_0(\rho) = [\sigma_0^2\{\rho^{|i-j|}\}_{1\le i\le j}]$, which is commonly observed in industrial & agricultural experiments, Econometrics, regression analysis (Chatterjee and Price, 2000). For the autocorrelated error structure $W_0(\rho)$, the combined error structure $\sigma_0^2 W_1$ as in (2.2) reduces to the following structure

$$\sigma_1^2 W_1 = \sigma_1^2 \begin{pmatrix} 1 & q\rho & q\rho^2 & \cdots & q\rho^{N-1} \\ q\rho & 1 & q\rho & \cdots & q\rho^{N-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q\rho^{N-1} & q\rho^{N-2} & q\rho^{N-3} & \cdots & 1 \end{pmatrix}$$

where $q = \dfrac{\sigma_0^2}{\sigma_1^2}$ and $\sigma_1^2 = (\sigma^2\pi^2/6 + \sigma_0^2)$ for Weibull distribution, $\sigma_1^2 = (\pi^2/6 + \sigma_0^2)$ for exponential distribution, $\sigma_1^2 = (\sigma_1^2 + \sigma_0^2)$ for gamma distribution and $\sigma_1^2 = (\delta_1^2 + \sigma_0^2)$ for lognormal distribution.

The above combined error structure $\sigma_1^2 W_1$ is complicated but it is approximately first-order autocorrelation structure $W_0(\rho)$ as in above (Das, Kim and Lee, 2015). The inverse of $\sigma_1^2 W_1$ does not exit in a general form with the parameters involved in it. To obtain an approximate inverse of it in a suitable form, it is assumed by Das, Kim and Lee (2015) that $q\rho^r = \rho_1^r$, for $r = 2, 3, ..., (N-1)$ where $\rho_1 = q\rho$. Note that, $0 < q < 1$ and $-1 < \rho < 1$, and the sign of $q\rho^r$ is the sign of ρ_1^r. This particular assumption is approximately true for smaller values of r. For $r = 1$, it is exactly true, but for $r > 1$, we consider a little lower value ρ_1^r than $q\rho^r$, in the population error dispersion structure which is completely unknown.

According to the above assumption, the approximated combined error structure turns to an autocorrelated structure $\sigma_1^2 W_1(\rho_1)$ with autocorrelation coefficient ρ_1, and the inverse of the approximated $\sigma_1^2 W_1$ is given as follow

$$(\sigma_1^2 W_1)^{-1} \approx (\sigma_1^2 W_1(\rho_1))^{-1} = \{\sigma_1^2(1-\rho_1^2)\}^{-1}[(1+\rho_1^2)I_N - \rho_1^2 B_0 - \rho_1 A_0]$$

where I_N is the $N \times N$ identity matrix, B_0 is the $N \times N$ matrix with elements $b_{11} = b_{NN} = 1$ and all other elements 0 (zeros), and A_0 is the $N \times N$ matrix with $a_{ij} = 1$ for $|i-j| = 1$ and all other elements 0 (zeros).

Simplified first-order rotatability conditions of (2.3) under the combined error structure $\sigma_1^2 W_1(\rho_1(\text{or } \rho))$ (for all the considered four lifetime distributions; for all $1 \le i, j \le k$) are

(i) $v_{0\cdot j} = \sum_{u=1}^{N} y_{uj} - \rho \sum_{u=2}^{N-1} y_{uj} = 0;\ 1 \le i, j \le k,$

(ii) $v_{i\cdot j} = \sum_{u=1}^{N} y_{ui} y_{uj} + \rho^2 \sum_{u=2}^{N-1} y_{uj} y_{uj} - \rho \left\{ \sum_{u=1}^{N-1} y_{uj} y_{(u+1)j} + \sum_{u=1}^{N-1} y_{(u+1)i} y_{uj} \right\} = 0;\ 1 \le i \ne j \le k,$

(iii) $v_{i\cdot i} = \{\sigma^2(1-\rho^2)\}^{-1}\left[\sum_{u=1}^{N} y_{ui}^2 + \rho^2\sum_{u=2}^{N-1} y_{ui}^2 - 2\rho\sum_{u=1}^{N-1} y_{ui}y_{(u+1)i}\right] = \lambda;\ 1 \le i \le k.$ (3.1)

A general method of RFORDs construction along with some other design examples are given in Das, Kim and Lee (2015).

The variance of the estimated response at y of a D-optimum robust first-order rotatable design (D-ORFOD) under the autocorrelated structure is

$$Var(\hat{x}_y)_0 = \sigma^2\left[\frac{1+\rho}{N-(N-2)\rho} + \frac{(1-\rho^2)r^2}{N+(N-2)\rho^2+2\rho(N-1)}\right] \text{ if } \rho > 0, \quad (3.2)$$

$$Var(\hat{x}_y)_0 = \sigma^2\left[\frac{1+\rho}{N-(N-2)\rho} + \frac{(1-\rho^2)r^2}{N+(N-2)\rho^2-2\rho(N-1)}\right] \text{ if } \rho < 0 \quad (3.3)$$

where $r^2 = \sum_{i=1}^{k} y_i^2$.

In case of autocorrelated error structure, D-ORFORDs are very difficult to construct, and yet they have been not developed best of our knowledge. So, some RFORDS have been developed by Das, Kim and Lee (2015). For a RFORD d with estimated variance $Var(\hat{x}_y)_d$, the efficiency ratio of the RFORD d is denoted by ER and is given by

$$ER\% = \frac{\max_{0\le r^2\le k}\{V(\hat{x}_y)_0\}}{\max_{0\le r^2\le k}\{V(\hat{x}_y)_d\}} \times 100, \quad (3.4)$$

where $V(\hat{x}_y)_0$ is the estimated variance a D-ORFOD under the autocorrelated structure as in (3.2) and (3.3).

Herein some examples of efficient RFORDs under autocorrelated error structure for the above four lifetime distributions are developed based on Hadamard matrices H_4, H_8 and H_{16}, where $H_4 = H_2 \otimes H_2$; $H_8 = H_2 \otimes H_4$; $H_{16} = H_2 \otimes H_8$, $H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\otimes$ denotes Kronecker product. The efficiency of the developed designs herein are compared with the designs developed by Das, Kim and Lee (2015). Higher order designs have not been considered herein as the design points will be large. Let $h_j^{(i)}$ be the j^{th} column of a Hadamard matrix of order i = 4, 8, 16. We have developed six efficient designs with k = 2, 3 and 4 factors for $\rho > 0$ and $\rho < 0$ with the same runs in comparison with the designs developed by Das, Kim and Park (2015). The current developed designs are $d_1(k = 2, N = 11, \rho > 0)$, $d_2(k = 3, N = 15, \rho > 0)$, $d_3(k = 4, N = 19, \rho > 0)$, $d_4(k = 2, N = 11, \rho < 0)$, $d_5(k = 3, N = 15, \rho < 0)$, $d_6(k = 4, N = 19, \rho < 0)$, which are displayed in Table 1 and Table 2, respectively. Similar six designs from Das, Kim and Lee (2015) are $D_1(k = 2, N = 11, \rho > 0)$, $D_2(k = 3, N = 15, \rho > 0)$, $D_3(k = 4, N = 19, \rho > 0)$, $D_4(k = 2, N = 11, \rho < 0)$, $D_5(k = 3, N = 15, \rho < 0)$, $D_6(k = 4, N = 19, \rho < 0)$, which are displayed in Table 3 and Table 4, respectively. Comparison of their efficiencies are displayed in the Tables 5 to 10, and in the Figures 1, 2 and 3. From Tables 5 to 10, and Figures 1 to

3, it is observed that the newly developed designs herein with the same number of factors and runs are more efficient than the designs developed by Das, Kim and Lee (2015).

4. First-order Correlated Lifetime Designs for A Special Compound Symmetry Correlation Structure

In case of a compound symmetry correlation structure, there are '*m*' groups each with '*n*' observations, and in each group there is an uniform correlation structure (Das, Kim and Park, 2015). This is an extension of inter-class correlation structure which is clearly explained in the article by Das, Kim and Park (2015). In the present context, we have considered a special type of compound symmetry correlation structure with two groups such that in the first group there is only the first observation, and the remaining group contains the rest $(N-1)$ observations. This situation is commonly observed when the machine is started initially, the first observation may be recorded with little more disturbance than the remaining others. As a result, the correlation between the first observation with the remaining is little different than the correlation between any two observations of the rest, excluding the first one. This is observed in practice in any production process, or in the measuring units with some instruments, etc. The first group may contain one or more observations. In the very sensitive cases, it may be only the first observation as the first group, and the rest others as the second group. This particular correlation structure is will illustrated in the article by Das and Mukherjee (2021).

Table 1: Newly developed efficient designs (d_1, d_2, d_3) for $\rho \geq 0$

Design	*No. of runs*	$\sum_{u=1}^{N} x_{ui}^2$	$\sum_{u=2}^{N-1} x_{ui}^2$	$\sum_{u=1}^{N} x_{ui} x_{(u+1)i}$	*ER%*
d_1 : $x_1 = (0, h_2^{(4)\prime}, 0, h_2^{(4)\prime}, 0)'$ $x_2 = (0, h_2^{(4)\prime}, 0, -h_2^{(4)\prime}, 0)'$	11	8 $(\neq 11)$	8 $(\neq 9)$	–6 $(\neq -10)$	$\dfrac{\left[\frac{(1+\rho)}{11-9\rho}+\frac{2(1-\rho^2)}{11+9\rho^2+20\rho}\right]}{\left[\frac{(1+\rho)}{11-9\rho}+\frac{2(1-\rho^2)}{8+8\rho^2+12\rho}\right]}$
d_2 : $x_1 = (0, h_4^{(4)\prime}, 0, -h_6^{(8)\prime}, 0)'$ $x_2 = (0, h_2^{(4)\prime}, 0, -h_8^{(8)\prime}, 0)'$ $x_3 = (0, h_3^{(4)\prime}, 0, +h_2^{(8)\prime}, 0)'$	15	12 $(\neq 15)$	12 $(\neq 13)$	–6 $(\neq -14)$	$\dfrac{\left[\frac{(1+\rho)}{15-13\rho}+\frac{3(1-\rho^2)}{15+13\rho^2+28\rho}\right]}{\left[\frac{(1+\rho)}{15-13\rho}+\frac{3(1-\rho^2)}{12+12\rho^2+12\rho}\right]}$
d_3 : $x_1 = (0, h_2^{(8)\prime}, 0, +h_8^{(8)\prime}, 0)'$ $x_2 = (0, h_8^{(8)\prime}, 0, -h_2^{(8)\prime}, 0)'$ $x_3 = (0, h_6^{(8)\prime}, 0, h_6^{(8)\prime}, 0)'$ $x_4 = (0, h_6^{(8)\prime}, 0, -h_6^{(8)\prime}, 0)'$	19	16 $(\neq 19)$	16 $(\neq 17)$	–10 $(\neq -18)$	$\dfrac{\left[\frac{(1+\rho)}{19-17\rho}+\frac{4(1-\rho^2)}{19+17\rho^2+36\rho}\right]}{\left[\frac{(1+\rho)}{19-17\rho}+\frac{3(1-\rho^2)}{16+16\rho^2+20\rho}\right]}$

Table 2: Newly developed efficient designs (d_4, d_5, d_6) for $\rho \le 0$

Design	*No. of runs*	$\sum_{u=1}^{N} x_{ui}^2$	$\sum_{u=2}^{N-1} x_{ui}^2$	$\sum_{u=1}^{N} x_{ui}x_{(u+1)i}$	*ER%*
d_4 : $x_1 = (0, h_3^{(4)\prime}, 0, h_3^{(4)\prime}, 0)'$ $x_2 = (0, h_3^{(4)\prime}, 0, -h_3^{(4)\prime}, 0)'$	11	8 ($\neq$ 11)	8 ($\neq$ 9)	2 ($\neq$ +10)	$\dfrac{\left[\frac{(1+\rho)}{11-9\rho}+\frac{2(1-\rho^2)}{11+9\rho^2-20\rho}\right]}{\left[\frac{(1+\rho)}{11-9\rho}+\frac{2(1-\rho^2)}{8+8\rho^2-4\rho}\right]}$
d_5 : $x_1 = (0, h_3^{(4)\prime}, 0, -h_3^{(8)\prime}, 0)'$ $x_2 = (0, h_2^{(4)\prime}, 0, h_5^{(8)\prime}, 0)'$ $x_3 = (0, h_4^{(4)\prime}, 0, h_7^{(8)\prime}, 0)'$	15	12 ($\neq$ 15)	12 ($\neq$ 13)	2 ($\neq$ 14)	$\dfrac{\left[\frac{(1+\rho)}{15-13\rho}+\frac{3(1-\rho^2)}{15+13\rho^2-28\rho}\right]}{\left[\frac{(1+\rho)}{15-13\rho}+\frac{3(1-\rho^2)}{12+12\rho^2-4\rho}\right]}$
d_6 : $x_1 = (0, h_3^{(8)\prime}, 0, +h_5^{(8)\prime}, 0)'$ $x_2 = (0, h_5^{(8)\prime}, 0, -h_3^{(8)\prime}, 0)'$ $x_3 = (0, h_7^{(8)\prime}, 0, h_7^{(8)\prime}, 0)'$ $x_4 = (0, h_7^{(8)\prime}, 0, -h_7^{(8)\prime}, 0)'$	19	16 ($\neq$ 19)	16 ($\neq$ 17)	6 ($\neq$ 18)	$\dfrac{\left[\frac{(1+\rho)}{19-17\rho}+\frac{4(1-\rho^2)}{19+17\rho^2-36\rho}\right]}{\left[\frac{(1+\rho)}{19-17\rho}+\frac{3(1-\rho^2)}{16+16\rho^2-12\rho}\right]}$

Table 3: Das, Kim and Lee (2015) developed designs (D_1, D_2, D_3) for $\rho \ge 0$

Design	*No. of runs*	$\sum_{u=1}^{N} x_{ui}^2$	$\sum_{u=2}^{N-1} x_{ui}^2$	$\sum_{u=1}^{N} x_{ui}x_{(u+1)i}$	*ER%*
D_1 : $x_1 = (0, h_2^{(4)\prime}, 0, h_4^{(4)\prime}, 0)'$ $x_2 = (0, h_4^{(4)\prime}, 0, h_2^{(4)\prime}, 0)'$	11	8 ($\neq$ 11)	8 ($\neq$ 9)	–4 ($\neq$ –10)	$\dfrac{\left[\frac{(1+\rho)}{11-9\rho}+\frac{2(1-\rho^2)}{11+9\rho^2+20\rho}\right]}{\left[\frac{(1+\rho)}{11-9\rho}+\frac{2(1-\rho^2)}{8+8\rho^2+8\rho}\right]}$
D_2 : $x_1 = (0, h_2^{(4)\prime}, 0, h_4^{(8)\prime}, 0)'$ $x_2 = (0, h_3^{(4)\prime}, 0, h_6^{(8)\prime}, 0)'$ $x_3 = (0, h_4^{(4)\prime}, 0, h_8^{(8)\prime}, 0)'$	15	12 ($\neq$ 15)	12 ($\neq$ 13)	–4 ($\neq$ –14)	$\dfrac{\left[\frac{(1+\rho)}{15-13\rho}+\frac{3(1-\rho^2)}{15+13\rho^2+28\rho}\right]}{\left[\frac{(1+\rho)}{15-13\rho}+\frac{3(1-\rho^2)}{12+12\rho^2+8\rho}\right]}$
D_3 : $x_1 = (0, h_2^{(8)\prime}, 0, h_4^{(8)\prime}, 0)'$ $x_2 = (0, h_4^{(8)\prime}, 0, h_2^{(8)\prime}, 0)'$ $x_3 = (0, h_6^{(8)\prime}, 0, h_8^{(8)\prime}, 0)'$ $x_4 = (0, h_8^{(8)\prime}, 0, h_6^{(8)\prime}, 0)'$	19	16 ($\neq$ 19)	16 ($\neq$ 17)	–8 ($\neq$ –18)	$\dfrac{\left[\frac{(1+\rho)}{19-17\rho}+\frac{4(1-\rho^2)}{19+17\rho^2+36\rho}\right]}{\left[\frac{(1+\rho)}{19-17\rho}+\frac{4(1-\rho^2)}{16+16\rho^2+16\rho}\right]}$

Table 4: Das, Kim and Lee (2015) developed designs (D_4, D_5, D_6) for $\rho \le 0$

Design	*No. of runs*	$\sum_{u=1}^{N} x_{ui}^2$	$\sum_{u=2}^{N-1} x_{ui}^2$	$\sum_{u=1}^{N} x_{ui} x_{(u+1)i}$	*ER%*
D_4 :					
$x_1 = (0, h_3^{(4)\prime}, 0, h_4^{(4)\prime}, 0)'$	11	8	8	0	$\dfrac{\left[\dfrac{(1+\rho)}{11-9\rho}+\dfrac{2(1-\rho^2)}{11+9\rho^2-20\rho}\right]}{\left[\dfrac{(1+\rho)}{11-9\rho}+\dfrac{2(1-\rho^2)}{8+8\rho^2}\right]}$
$x_2 = (0, h_4^{(4)\prime}, 0, h_3^{(4)\prime}, 0)'$		($\neq$ 11)	($\neq$ 9)	($\neq$ 10)	
D_5 :					
$x_1 = (0, h_2^{(4)\prime}, 0, h_7^{(8)\prime}, 0)'$	15	12	12	0	$\dfrac{\left[\dfrac{(1+\rho)}{15-13\rho}+\dfrac{3(1-\rho^2)}{15+13\rho^2-28\rho}\right]}{\left[\dfrac{(1+\rho)}{15-13\rho}+\dfrac{3(1-\rho^2)}{12+12\rho^2}\right]}$
$x_2 = (0, h_3^{(4)\prime}, 0, h_4^{(8)\prime}, 0)'$		($\neq$ 15)	($\neq$ 13)	($\neq$ 14)	
$x_3 = (0, h_4^{(4)\prime}, 0, h_3^{(8)\prime}, 0)'$					
D_6 :					
$x_1 = (0, h_3^{(8)\prime}, 0, h_7^{(8)\prime}, 0)'$	19	16	16	4	$\dfrac{\left[\dfrac{(1+\rho)}{19-17\rho}+\dfrac{4(1-\rho^2)}{19+17\rho^2-36\rho}\right]}{\left[\dfrac{(1+\rho)}{19-17\rho}+\dfrac{4(1-\rho^2)}{16+16\rho^2-8\rho}\right]}$
$x_2 = (0, h_7^{(8)\prime}, 0, h_3^{(8)\prime}, 0)'$		($\neq$ 19)	($\neq$ 17)	($\neq$ 18)	
$x_3 = (0, h_4^{(8)\prime}, 0, h_5^{(8)\prime}, 0)'$					
$x_4 = (0, h_5^{(8)\prime}, 0, h_4^{(8)\prime}, 0)'$					

Under the above special situations only with two groups, it is assumed that the N observational errors e_1, e_2, ..., e_N have the same variance σ_0^2, and $\dfrac{Cov(e_i, e_j)}{\sigma_0^2} = \rho^*$; $i \neq j = 2, 3, ..., N$; also $\dfrac{Cov(e_1, e_i)}{\sigma_0^2} = \rho_1^*$; $i = 2, 3, ..., N$. Therefore, Dis(e) = $\sigma_0^2 W_0(\rho_1^*, \rho^*)$ say, and it is given by

$$\sigma_0^2 W_0(\rho_1^*, \rho^*) = \sigma_0^2 \begin{pmatrix} 1 & \rho_1^* & \rho_1^* & \cdots & \rho_1^* \\ \rho_1^* & 1 & \rho^* & \cdots & \rho^* \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \rho_1^* & \rho^* & \rho^* & \cdots & 1 \end{pmatrix}$$

For the above special compound symmetry dispersion matrix $\sigma_0^2 W_0(\rho_1^*, \rho^*)$, the combined error structure $\sigma_1^2 W_1$ as in (2.2) reduces to $\sigma_1^2 W_0(\rho_1, \rho)$, where $\rho_1 = q\rho_1^*$, $\rho = q\rho^*$, $q = \dfrac{\sigma_0^2}{\sigma_1^2}$ and $\sigma_1^2 = (\sigma^2\pi^2/6 + \sigma_0^2)$ for Weibull distribution, $\sigma_1^{\,2} = (\pi^2/6 + \sigma_0^2)$ for exponential distribution, $\sigma_1^{\,2} = (\alpha_1^2 + \sigma_0^2)$ for gamma distribution and $\sigma_1^2 = (\delta_1^2 + \sigma_0^2)$ for lognormal distribution. Therefore,

Table 5: *ER*% of the designs d_1 and D_1 when $\rho \geq 0$

ρ	*ER*% of d_1	*ER*% of D_1
0.0	80.00000	80.00000
0.1	80.73013	78.39230
0.2	82.52096	78.84199
0.3	85.03797	80.80545
0.4	87.96580	83.81011
0.5	90.99164	87.38526
0.6	93.82460	91.05691
0.7	96.23336	94.39488
0.8	98.07823	97.08028
0.9	99.31954	98.95102

$$\{\sigma_1^2 W_0(\rho_1,\rho)\}^{-1} = (\sigma_1^2)^{-1}\begin{pmatrix} a & b & b & \cdots & b \\ b & c & d & \cdots & d \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b & d & d & \cdots & c \end{pmatrix},$$

where, $a = \dfrac{[1+(N-2)\rho]}{[1+(N-2)\rho-(N-1)\rho_1^2)]}$, $b = \dfrac{-\rho_1}{[1+(N-2)\rho-(N-1)\rho_1^2)]}$,

$c = \dfrac{[1+(N-3)\rho-(N-2)\rho_1^2)]}{(1-\rho)[1+(N-2)\rho-(N-1)\rho_1^2)]}$, and $d = \dfrac{(\rho_1^2-\rho)}{(1-\rho)[1+(N-2)\rho-(N-1)\rho_1^2)]}$.

First-order Rotatability Conditions

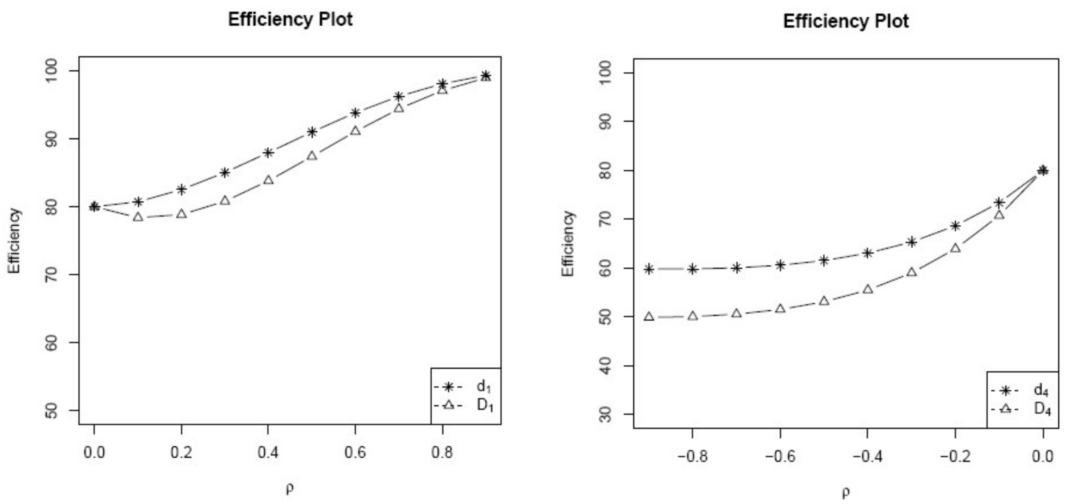
Simplified first-order rotatability conditions of (2.3) under the combined error structure $\sigma_0^2 W_0(\rho_1, \rho)$ (for all the considered four lifetime distributions; for all $1 \leq i, j \leq k$) are

(i) $\{a + (N-1)b\}\, y_{1j} + \{(b + c) + (N-2)d\} \sum_{u=2}^{N} y_{uj} = 0$, for all j

(ii) $a y_{1i} y_{1j} + (c-d) \sum_{u=2}^{N} y_{ui} y_{uj} + b\left\{\left(\sum_{u=2}^{N} y_{ui}\right) y_{1j} + \left(\sum_{u=2}^{N} y_{uj}\right) y_{1i}\right\} + d\left(\sum_{u=2}^{N} y_{ui}\right)\left(\sum_{u=2}^{N} y_{uj}\right) = 0$, for all $i \neq j$

(iii) $\sigma_1^{-2}\left[a y_{1i}^2 + (c-d)\sum_{u=2}^{N} y_{ui}^2 + 2b\left\{\left(\sum_{u=2}^{N} y_{ui}\right) y_{1i}\right\} + d\left(\sum_{u=2}^{N} y_{ui}\right)^2\right] = \lambda$, for all i (4.1)

where the values of a, b, c and d are given above.



(a) **(b)**

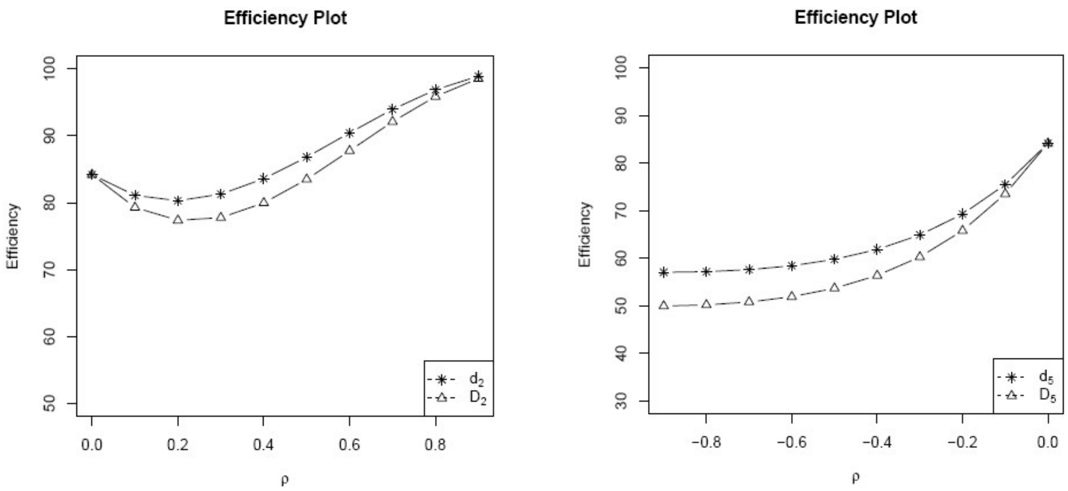
Figure 1: The precision of the designs (a) (d_1, D_1) and (b) (d_4, D_4) with k = 2, N = 11 for –0.9 ≤ ρ ≤ 0.9 based on average r^2 (Table 5 and Table 8)

Table 6: *ER*% of the designs d_2 and D_2 when ρ ≥ 0

ρ	*ER% of* d_2	*ER% of* D_2
0.0	84.21053	84.21053
0.1	81.09613	79.29124
0.2	80.30303	77.37226
0.3	81.28186	77.76786
0.4	83.59965	79.97357
0.5	86.82171	83.50119
0.6	90.45562	87.77589
0.7	93.97301	92.12634
0.8	96.90249	95.88898
0.9	98.93951	98.57698

Design Construction Method

From an usual (independent errors) first-order rotatable design (FORD) with k factors and N runs, one can construct a RFORD under the special compound symmetry structure $\sigma_1^2 W_0(\rho_1, \rho)$ (for all the considered four lifetime distributions, and for all values of ρ_1, ρ), just adding one central point at the beginning of each factor of the FORD. The resultant design will be a RFORD under this special structure with k factors and (N + 1) runs. Note that the resultant design will satisfy (4.1) always for all values of ρ_1, ρ and for all the considered four lifetime distributions. Therefore, the derived designs are robust (free of ρ_1, ρ values) and invariant of the considered four lifetime



(a) (b)

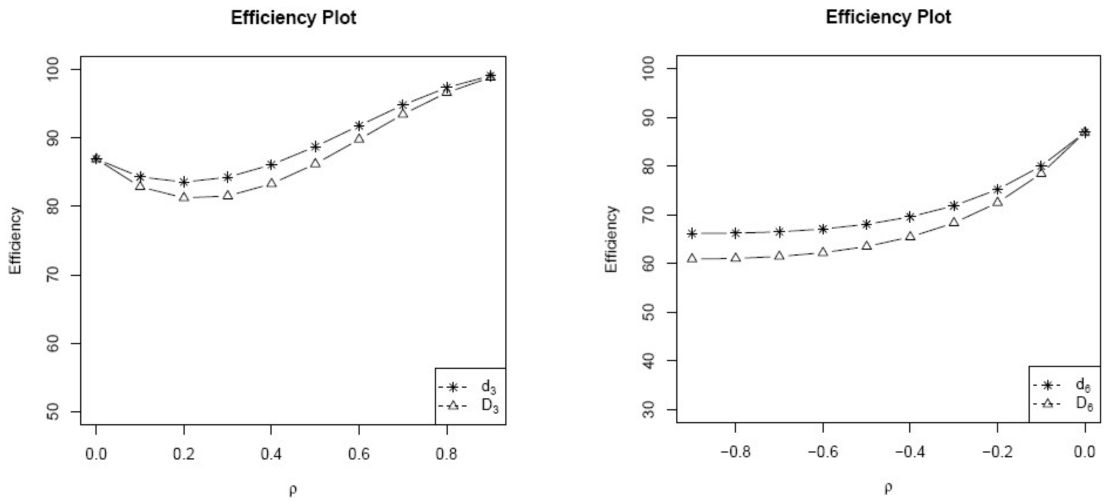
Figure 2: The precision of the designs (a) (d_2, D_2) and (b) (d_5, D_5) with k = 3, N = 15 for $-0.9 \le \rho \le 0.9$ based on average r^2 (Table 6 and Table 9)

Table 7: *ER*% of the designs d_3 and D_3 when $\rho \ge 0$

ρ	*ER% of* d_3	*ER% of* D_3
0.0	86.95652	86.95652
0.1	84.32475	82.87951
0.2	83.57629	81.25819
0.3	84.28250	81.53672
0.4	86.12171	83.32108
0.5	88.77005	86.23377
0.6	91.83580	89.81285
0.7	94.86062	93.48770
0.8	97.40054	96.66538
0.9	99.14627	98.89540

distribution. The variance function of a RFORD is similarly given by (2.4). An example of a RFORD (d) with three factors (k = 3) and five runs (N = 5) for the special compound symmetry structure $\sigma_1^2 W_0(\rho_1, \rho)$, following the above method is given as follow.

d :	1	2	3	4	5
y_1	0	1	–1	1	–1
y_2	0	1	1	–1	–1
y_3	0	1	–1	–1	1



(a) **(b)**

Figure 3: The precision of the designs (a) (d_3, D_3) and (b) (d_6, D_6) with $k = 4$, $N = 19$ for $-0.9 \leq \rho \leq 0.9$ based on average r^2 (Table 7 and Table 10)

Table 8: *ER*% of the designs d_4 and D_4 when $\rho \leq 0$

ρ	*ER% of* d_4	*ER% of* D_4
0.0	80.00000	80.00000
–0.1	73.39873	70.75306
–0.2	68.67470	63.93443
–0.3	65.34914	58.99865
–0.4	63.06028	55.50239
–0.5	61.53846	53.09735
–0.6	60.58394	51.51515
–0.7	60.04895	50.55131
–0.8	59.82405	50.05086
–0.9	59.82793	49.89662

5. Concluding Remarks

The article includes four fundamental lifetime distributions such as lognormal, exponential, gamma and Weibull as the response lifetime distribution. Moreover, it considers that the experimental errors are autocorrelated and a special compound symmetry structure. For handling lifetime response heterogeneity, regressor controllable variables are considered. First-order location-scale lifetime correlated models are taken in the article. Mixed linear logarithm of lifetime correlated first-order

Table 9: *ER*% of the designs d_5 and D_5 when ρ ≤ 0

ρ	*ER% of* d_5	*ER% of* D_5
0.0	84.21053	84.21053
–0.1	75.52405	73.55485
–0.2	69.31106	65.82278
–0.3	64.91647	60.28344
–0.4	61.85731	56.37910
–0.5	59.78022	53.69128
–0.6	58.42697	51.90840
–0.7	57.60847	50.79906
–0.8	57.18570	50.19128
–0.9	57.05594	49.95687

Table 10: *ER*% of the designs *d*6 and *D*6 when ρ ≤ 0

ρ	*ER% of* d_6	*ER% of* D_6
0.0	86.95652	86.95652
–0.1	80.04426	78.48945
–0.2	75.22124	72.51908
–0.3	71.87756	68.33839
–0.4	69.59009	65.44755
–0.5	68.06283	63.49206
–0.6	67.08673	62.21889
–0.7	66.51263	61.44598
–0.8	66.23300	61.04129
–0.9	66.16962	60.90823

models are considered herein with two random variables, where one is connected with the lifetime response distribution, and the other is related with the error component. First-order efficient rotatable designs are derived herein with autocorrelated errors along with an approximated composite error dispersion matrix. For a special compound symmetry error structure, a method of RFORDs is developed herein. The designs derived herein do not depend on the response four lifetime distributions as well as the values of correlation coefficients of the error structure. Therefore, these designs are termed as invariant robust.

Real lifetime observations reveal that they are heteroscedatic along with lognormal, gamma, Weibull, exponential distributions (Lawless, 1982; Das, Kim and Park, 2015; Das, 2013; Das and

Lee, 2009). Myers *et al*. (2002, p.128) illustrated that in *industrial production processes* experimental units are not independent at times *by design*, which incorporates correlation among observations via a *repeated measures* scenario as in *split plot* design. From practical view point, the present paper has considered correlated first-order non-linear models, $\hat{t}_{0u} = \exp(\hat{\beta}_0 + \sum_{i=1}^{k} \hat{\beta}_i y_{ui})$ for some useful lifetimes distributions.

The developed designs herein can be used to interpret the optimal operating settings which attains the target mean value, while reducing the variance. For this purpose, Myers and Carter (1973) adopted dual response surface (DRS) approach, while Nelder and Lee (1991) applied joint generalized linear models (JGLMs). Therefore, the current developed designs can be applied for both the DRS and JGLMs approaches appropriately.

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